Quantifying Volatility in VaR Models

量化 VaR 模型中的波动率

Three Common Deviations From Normality

- Three common deviations from normality that are problematic in modeling risk result from asset returns that are **fat-tailed**, **skewed**, or **unstable**.

Figure 1: Illustration of Fat-Tailed and Normal Distributions

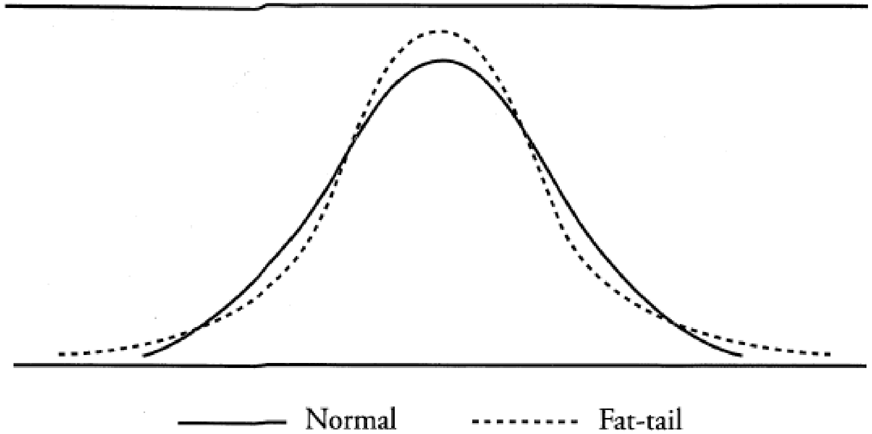
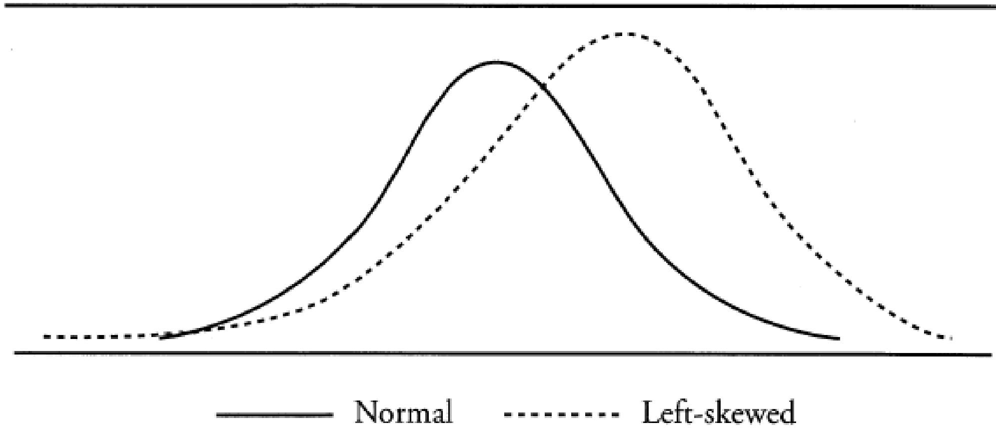


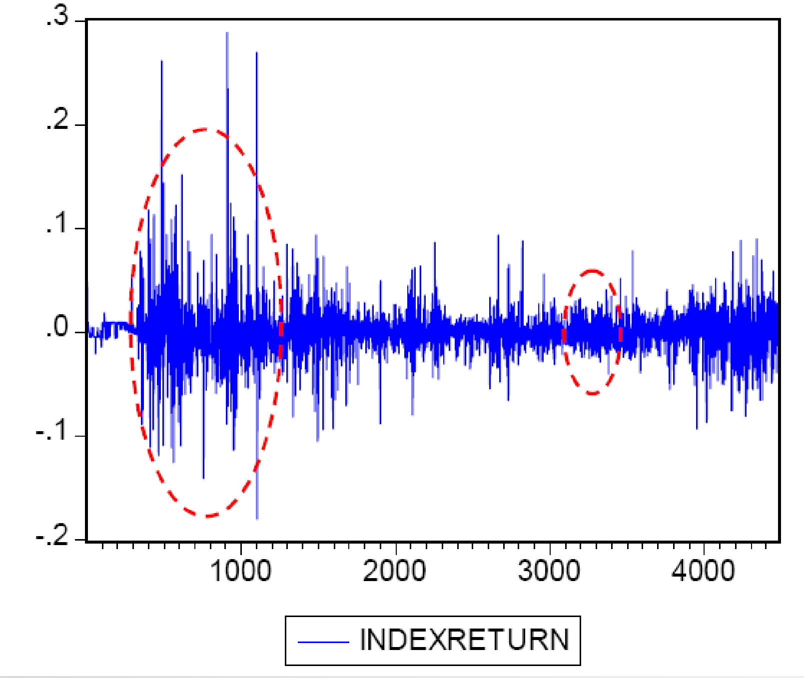
Figure 2: Left-Skewed and Normal Distributions



In modeling risk, a number of assumptions are necessary. If the parameters of the model are **unstable**, they are not constant but vary over time. For example, if interest rate , inflation, and market premiums are changing over time, this will affect the volatility of the returns going forward.

Deviations From The Normal Distribution

- The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time.
  - If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an **unconditional distribution** of asset returns.
  - However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a **conditional distribution**.

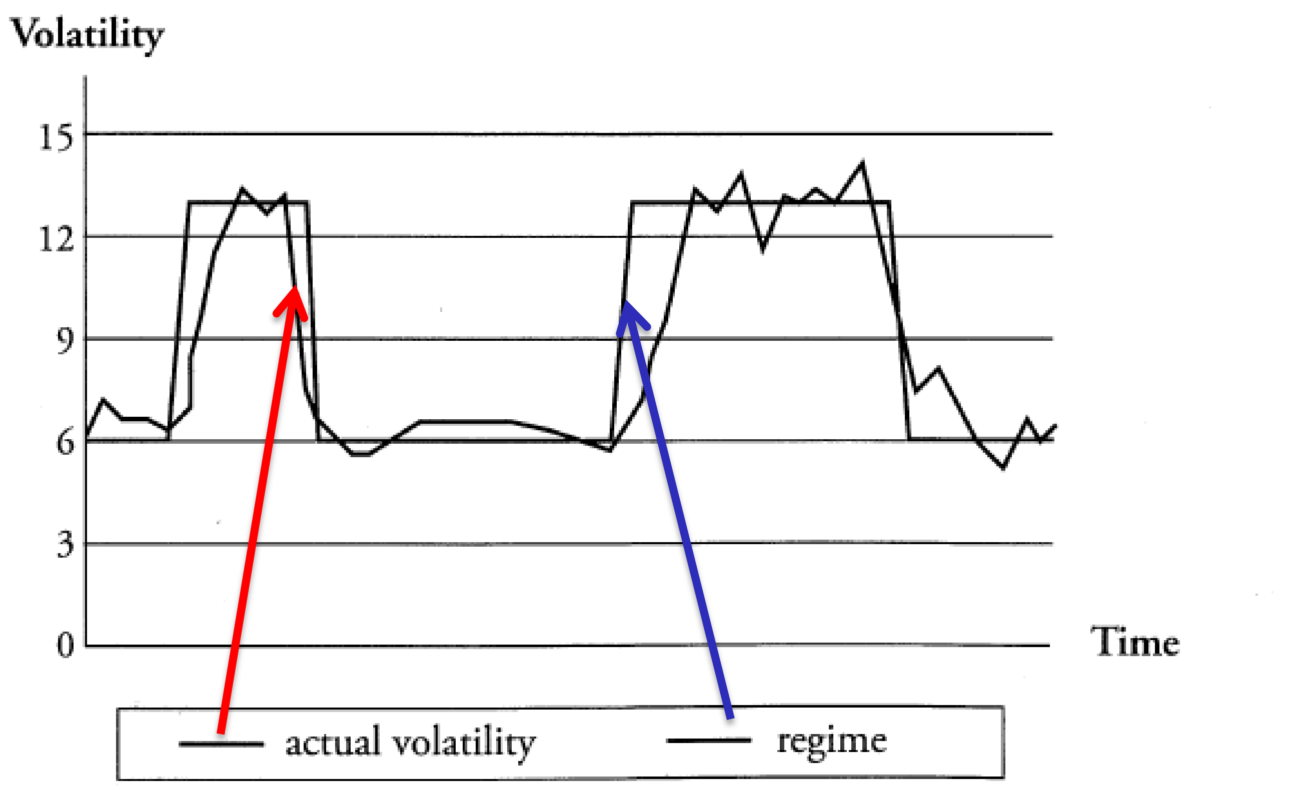


Volatility Clustering

Market Regimes (市场机制) and Conditional Distribution

- A **regime-switching volatility** model assumes different market regimes exist with high or low volatility. The conditional distributions of returns are always normal with a constant mean but either have a high or low volatility.

Figure 3: Actual Conditional Return Volatility Under Market Regimes



Value at Risk

- A value at risk (VaR) method for estimating risk is typically either a **historical-based approach** or an **implied-volatility-based approach**. Under the historical-based approach, the shape of the conditional distribution is estimated based on historical time series data.
- **Historical-based approaches** typically fall into three sub-categories: **parametric**, **nonparametric**, and **hybrid**.

1. The **parametric approach** requires specific assumptions regarding the asset returns distribution. A parametric model typically assumes asset returns are normally distributed with time-varying volatility. The most common example of the parametric method in estimating future volatility is based on calculating historical variance or standard deviation using "mean squared deviation." For example, the following equation is used to estimate future variance based on a window of the *K* most recent returns data.

$$\sigma_t^2 = \left(r^2_{t-K,t-K+1} + \cdots + r^2_{t-3,t-2} + r^2_{t-2,t-1} + r^2_{t-1,t} \right) \Big/ K$$

2. The **nonparametric approach** is less restrictive in that there are no underlying assumptions of the asset returns distribution. The most common nonparametric approach using the historical simulation method.
3. As the name suggests, the **hybrid approach** combines techniques of both parametric and nonparametric methods to estimate volatility using historical data.

- The **implied-volatility-based approach** uses derivative pricing models such as the Black -Scholes-Merton option pricing model to estimate an implied volatility based on current market data rather than historical data.

Parametric Approaches for VaR

① The **historical standard deviation** approach assumes all *k* returns in the window are equally weighted.

$$\sigma_t^2 = \left(r^2_{t-K,t-K+1} + \cdots + r^2_{t-3,t-2} + r^2_{t-2,t-1} + r^2_{t-1,t} \right) \Big/ K$$

② EWMA Model (RiskMetrics)

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2$$

$$\sigma_t^2 = (1-\lambda)\left(\lambda^0 r^2_{t-1,t} + \lambda^1 r^2_{t-2,t-1} + \lambda^2 r^2_{t-3,t-2} + \cdots + \lambda^N r^2_{t-N-1,t-N} \right)$$

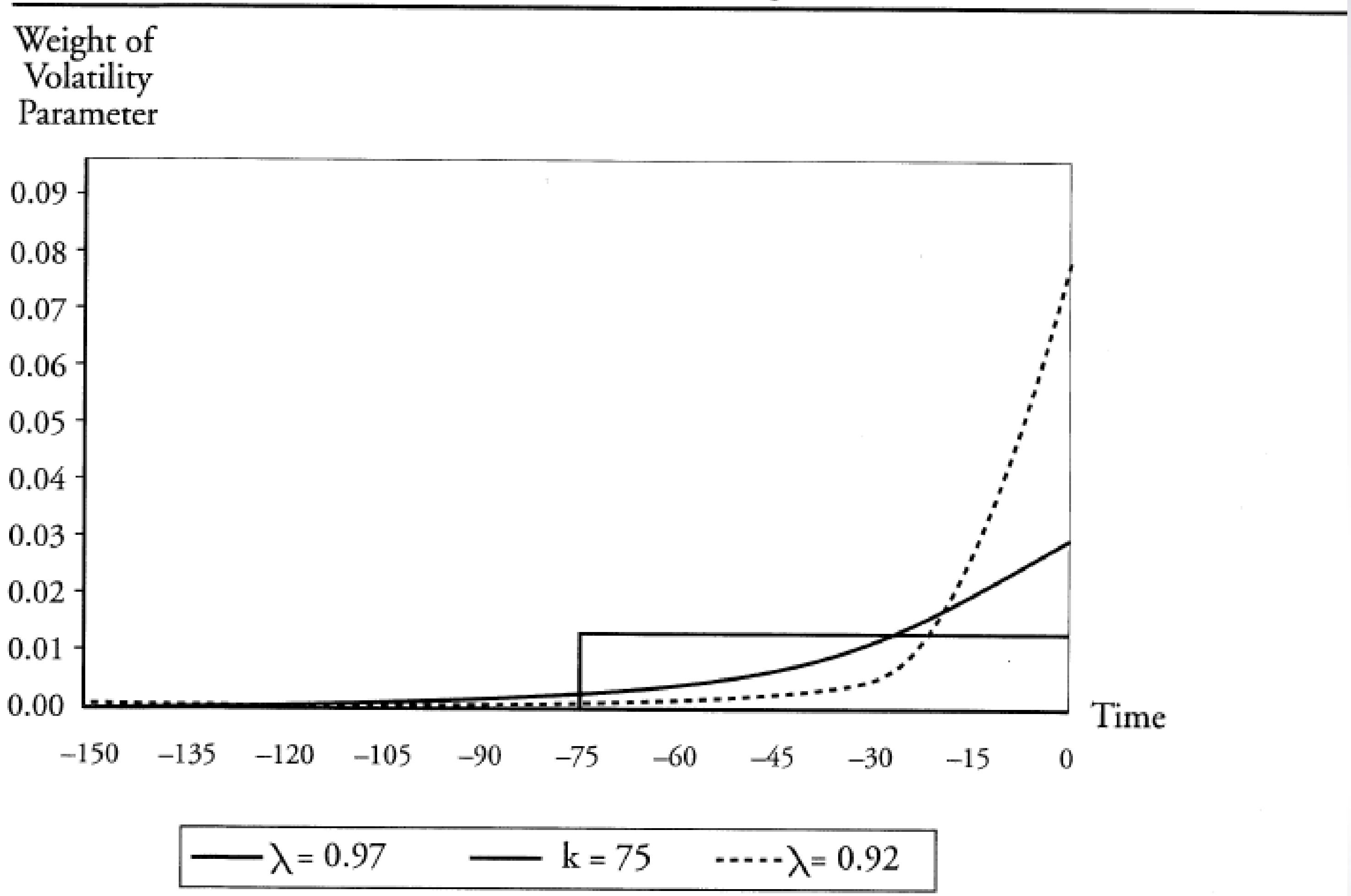
③ GARCH (1.1) Model

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Parametric Approaches for VaR

- The figure of EWMA model: λ is called decay factor.

Figure 4: Comparison of Exponential Smoothing and Historical Standard Deviation



Parametric Approaches for VaR

- Figure 5 summarizes the most recent weights for the volatility parameters using the three approaches used in Figure 4. Parameter λ values of 0.92 and 0.97 are used for the example of the RiskMetrics® approaches in Figure 4.

Figure 5: Summary of RiskMetrics® and Historical Standard Deviation Calculations

	Weight of Volatility Parameter		
	$(1-\lambda)\lambda^t$	$1/k$	$(1-\lambda)\lambda^t$
t	$\lambda = 0.97$	$k = 75$	$\lambda = 0.92$
0	0.0300	0.0133	0.0800
–1	0.0291	0.0133	0.0736
–2	0.0282	0.0133	0.0677
–3	0.0274	0.0133	0.0623
–4	0.0266	0.0133	0.0573

Nonparametric Approaches for VaR

① Historical Simulation Method

- The six lowest returns for an estimation window of 100 days (K = 100) are listed in Figure 6. Under the historical simulation, all returns are weighted equally based on the number of observations in the estimation window (1/K). Thus, in this example, each return has a weight of 1/100, or 0.01.

VaR(5%)

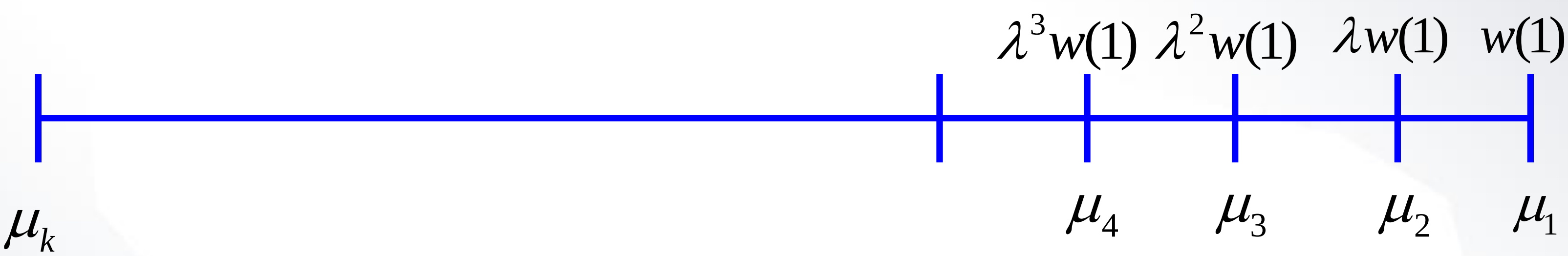
Figure 6: Historical Simulation Example

Six Lowest Returns	*Historical Simulation Weight*	*HS Cumulative Weight*
–4.70%	0.01	0.0100
–4.10%	0.01	0.0200
–3.70%	0.01	0.0300
–3.60%	0.01	0.0400
–3.40%	0.01	0.0500
–3.20%	0.01	0.0600

Nonparametric Approaches for VaR

② Hybrid Approach

- The hybrid approach uses historical simulation to estimate the percentiles of the return and weights that decline exponentially (similar to GARCH or RiskMetrics®).



$$w_{(1)} + \lambda w_{(1)} + \ldots + \lambda^{n-1} w_{(1)} = 1 \rightarrow w_{(1)} = (1-\lambda)/1-\lambda^k$$

Assign Weights

$$[(1-\lambda)/(1-\lambda^K)],\ [(1-\lambda)/(1-\lambda^K)]\lambda^1, \ldots,\ [(1-\lambda)/(1-\lambda^K)]\lambda^{K-1}$$

② Hybrid Approach

Figure 7: Hybrid Example Illustrating Six Lowest Returns (where K = 100 and λ = 0.96)

Rank	*Six Lowest Returns*	*Number of Past Periods*	*Hybrid Weight*	*Hybrid Cumulative Weight**
1	–4.70%	2	0.0391	0.0391
2	–4.10%	5	0.0346	0.0736
3	–3.70%	55	0.0045	0.0781
4	–3.60%	25	0.0153	0.0934
5	–3.40%	14	0.0239	0.1173
6	–3.20%	7	0.0318	0.1492

*Cumulative weights are slightly affected by rounding error.

② Hybrid Approach

Example: Calculating VaR using the hybrid approach

Using the information in Figure 7, **calculate** the initial VaR at the 5th percentile using the hybrid approach.

Answer:

The lowest and second lowest returns have cumulative weights of 3.91% and 7.36%, respectively. Therefore, we must interpolate to obtain the 5% VaR percentile. The point halfway between the two lowest returns is interpolated as –4.40% [(–4.70% + –4.10%) / 2] with a cumulative weight of 5.635% calculated as follows: (7.36% + 3.91%) / 2. Further interpolation is required to find the 5th percentile VaR level somewhere between –4.70% and –4.40%.

For the initial period represented in Figure 7, the 5% VaR using the hybrid approach is calculated as:

$$4.7\% - (4.70\% - 4.40\%)[(0.05 - 0.03910) / (0.05635 - 0.03910)]$$
$$= 4.70\% - 0.3\%(0.63188) = 4.510\%$$

Nonparametric Approaches for VaR

③ Multivariate Density Estimation (MDE)

- Conditional volatility for each market state or regime is calculated as follows:

$$\sigma_t^2 = \sum_{i=1}^{K} \omega(x_{t-i}) r_{t-i}^2$$

Where:

X_{t-i} = the vector of relevant variables describing the market state or regime at time t-i

$\omega(X_{t-i})$ = the weight used on observation t-i.

The **kernel** function, $\omega(X_{t-i})$, is used to measure the relative weight in terms of "near" or "distance" from the current state. The MDE model is very flexible in identifying dependence on state variables.

Some examples of relevant state variables are interest rate volatility dependent on the level of interest rates or the term structure of interest rates, equity volatility dependent on implied volatility, and exchange rate volatility dependent on interest rate spreads.

Mean Reversion and Long Time Horizons

- If mean reversion exists, the long horizon risk (and the resulting VaR) is smaller than the square root of volatility.

EXAMPLE 3.3: FRM EXAM 2002—QUESTION 2

Assume we calculate a one-week VAR for a natural gas position by rescaling the daily VAR using the square-root rule. Let us now assume that we determine the *true* gas price process to be mean-reverting and recalculate the VAR.

Which of the following statements is true?

a. The recalculated VAR will be less than the original VAR.
b. The recalculated VAR will be equal to the original VAR.
c. The recalculated VAR will be greater than the original VAR.
d. There is no necessary relation between the recalculated VAR and the original VAR.

Backtesting VaR Methodologies

- **Backtesting** is the process of comparing losses predicted by the value at risk (VaR) model to those actually experienced over the sample testing period. If a model were completely accurate, we would expect VaR to be exceeded (this is called an *exception)* with the same frequency predicted by the confidence level used in the VaR model. In other words, the probability of observing a loss amount greater than VaR is equal to the significance level (x%). This value is also obtained by calculating one minus the confidence level.
- *For example*, if a VaR of $10 million is calculated at a 95% confidence level, we expect to have exceptions (losses exceeding $10 million) 5% of the time. If exceptions are occurring with greater frequency, we may be underestimating the actual risk. If exceptions are occurring less frequently, we may be overestimating risk.

Example

1. Fat-tailed asset return distributions are most likely the result of time-varying:
   A. volatility for the unconditional distribution.
   B. means for the unconditional distribution.
   C. volatility for the conditional distribution.
   D. means for the conditional distribution.

Example

2. The problem of fat tails when measuring volatility is least likely:
   A. in a regime-switching model.
   B. in an unconditional distribution.
   C. in a historical standard deviation model.
   D. in an exponential smoothing model.

Example

3. The lowest six returns for a portfolio are shown in the following table.

Six lowest returns with hybrid weightings			
	Six Lowest Returns	*Hybrid Weight*	*Hybrid Cumulative Weight*
1	–4.10%	0.0125	0.0125
2	–3.80%	0.0118	0.0243
3	–3.50%	0.0077	0.0320
4	–3.20%	0.0098	0.0418
5	–3.10%	0.0062	0.0481
6	–2.90%	0.0027	0.0508

What will the 5% VaR be under the hybrid approach? Assume each observation is a random event with 50% to the left and 50% to the right of each observation.

A. -3.10%.

B. -3.04%.

C. -2.96%.

D. -2.90%.

Example

C The fifth and sixth lowest returns have cumulative weights of 4.81% and 5.08%, respectively. The point halfway between these two returns is interpolated as –3.00% with a cumulative weight of 4.945%, calculated as follows: (4.81% + 5.08%) / 2. Further interpolation is required to find the 5th percentile VaR level with a return somewhere between –3.00% and –2.90%. The 5% VaR using the hybrid approach is calculated as:

3.00% – (3.00% – 2.90%)[(0.05 – 0.04945) / (0.0508 – 0.04945)]
= 3.00% – 0.10%(0.0005 / 0.00135) = 2.96%

Notice that the answer has to be between –2.90% and –3.00%, so –2.96% is the only possible answer.

$-3\% \longrightarrow 4.945\%$

$x \longrightarrow 5\%$

$2.9\% \longrightarrow 5.08\%$

$$\frac{3\% - x}{3\% - 2.9\%} = \frac{5\% - 4.945\%}{5.08\% - 4.945\%}$$

恭祝大家

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